Pergamon

J. Appl. Maths Mechs, Vol. 61, No. 6, pp. 987–994, 1997

0021–8928/97 $24.00+0.00

PII: S0021–8928(97)00127-5

DEFINING RELATIONS FOR A VISCOELASTIC MEDIUM WITH MICROROTATIONS†

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(*Received* 10 *November* 1994)

The results of [1, 2] are extended to the case of a Cosserat medium with a memory (the force stress tensor and the couple stress tensor depend on the history of deformations and rotations of a particle in the medium). In the linear approximation the defining relations have the form of convolutions with some relaxation kernels with respect to time. Restrictions for the kernels are obtained, which follow from the general principles of thermodynamics. The propagation of weak perturbations is studied. The general functional form of the kernels corresponding to experimental data on the viscoelasticity of rock formations is given. © 1998 Elsevier Science Ltd. All rights reserved.

1. To a large extent we shall use the notation of the earlier papers [1, 2]. We denote the time in an inertial frame of reference by t and we denote the Cartesian coordinates by x. The Latin subscripts correspond to the coordinates and take the values 1, 2 and 3. Unless otherwise stated summation is carried out over repeated indices.

Suppose that a particle of the medium, having coordinates x_{i0} at time t_0, moves to a point with coordinates

$$x_{i1} = X_i(t_1, t_0, x_{i0}) \tag{1.1}$$

at time t_1. The corresponding complete rotation of the particle is given by the matrix $G(t_1, t_0, x_{i0}) \in SO(3)$ in the selected Cartesian coordinates. It is obvious that

$$X_i(t_0, t_0, x_{i0}) = x_{i0}, \quad G(t_0, t_0, x_{i0}) = 1$$

We recall that any element G of $SO(3)$ can be represented as

$$G = \exp(\omega_i \tau_i) \tag{1.2}$$

where $\omega_i = \omega_i(G)$ are real numbers, and τ_i are basis elements of the Lie algebra $so(3)$ such that if only one of the numbers φ_i in (1.2) is non-zero, then the matrix g defines a rotation by φ_i about the corresponding axis.

We define the velocity field and the field of angular velocities

$$v_i(t, x_j) = \partial_t X_i(t, t_0, x_j)\big|_{t_0=t}$$

$$\Omega(t, x_j) = \partial_t G(t, t_0, x_j)\big|_{t_0=t} \in so(3)$$

The components of the angular velocity can be defined in terms of the field of angular velocities,

$$\Omega_i(t, x_j) = \mathrm{Tr}(\tau_i \Omega(t, x_j))$$

The matrices $F \in G_+L(3)$ with elements

$$F_{ij}(t_1, t_0, x_{k0}) = \frac{\partial}{\partial x_{j0}} X_i(t_1, t_0, x_{k0}) \tag{1.3}$$

can be represented as $F = OD$, where $O \in SO(3)$, $D = D^T > 0$. It follows that the rotation and

†*Prikl. Mat. Mekh.* Vol. 61, No. 6, pp. 1023–1030, 1997.

deformation matrices

$$O = O(t_1, t_0, x_{k0}), \quad D = D(t_1, t_0, x_{k0})$$

in the fixed system of coordinates can be defined in terms of (1.3).

The relative rotation of the particle in the time interval between t_0 and t_1 is given by the matrix

$$R(t_1, t_0, x_{k0}) = O^{-1}(t_1, t_0, x_{k0}) G(t_1, t_0, x_{k0})$$

Let $\rho = \rho(t, x_i)$ be the mass density and $T = T(t, x_i)$ the absolute temperature of the medium. The following equations of continuity, momentum, angular momentum and energy are satisfied [2]

$$\frac{d\rho}{dt} + \rho v_{i,i} = 0, \quad \rho \frac{dv_i}{dt} = p_{ij,j} + f_i \tag{1.4}$$

$$\frac{d}{dt}(\varepsilon_{ijk} x_j \rho v_k + J\Omega_i) + (\varepsilon_{ijk} x_j \rho v_k + J\Omega_i) v_{l,l} = (\varepsilon_{ikl} x_k p_{lj} + \pi_{ij})_{,j} + \varepsilon_{ijk} x_j f_k + m_i + M_i \tag{1.5}$$

$$\frac{d}{dt}(K + \rho U) + (K + \rho U) v_{i,i} = (v_i p_{ij})_{,j} + (\Omega_i \pi_{ij})_{,j} + f_i v_i + m_i \Omega_i - q_{i,i} + \varepsilon \tag{1.6}$$

$$K = \frac{1}{2}\rho v_i v_i + \frac{1}{2} J \Omega_i \Omega_i, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i}$$

Here p_{ij} are the components of the force stress tensor, π_{ij} are the components of the symmetric tensor of the moment (or couple) stresses, f_i are the components of the external volume forces, m_i are the components of the momentum of the external forces, M_i are the components of the moment of internal forces, J is the density of the moment of inertia (having dimensions of mass $\times$ length^{-1}), which we shall assume to be constant, K is kinetic energy density, U is the internal energy of the particle per unit mass, q_i are the components of the energy flux vector, and ε is the heat generated per unit volume.

We shall use the Clausius–Duhem inequality [3, 4]

$$\int_{-\infty}^{+\infty} \sigma \, dt \geqslant 0, \quad \sigma = \rho \frac{ds}{dt} - T^{-1}\varepsilon + (q_i T^{-1})_{,i} \tag{1.7}$$

where σ is the entropy production in a particle of the medium and s is the entropy in a particle of the medium per unit mass. The integration in (1.7) is carried out over the states of the particle. It is assumed that the medium remains in the same state of rest as $t \to \pm\infty$.

To close the dynamical system we need to give specific expressions for p_{ij}, π_{ij}, q_i, M_i, U.

Suppose that the state of the medium is a stress-free state of rest. We introduce the notation

$$D^0(t, x_i) = D(-\infty, t, x_i), \quad R^0(t, x_i) = R(-\infty, t, x_i)$$

Suppose that the internal energy U of a particle of the medium depends on the parameters T, D^0_{ij}, R^0_{ij}. No dependence on ρ needs to be introduced because the first equation implies the relationship

$$\rho(t_1, x_{i1}) \operatorname{Det} D(t_1, t_0, x_{i0}) = \rho(t_0, x_{i0}) \tag{1.8}$$

for the points x_{i0}, x_{i1} connected by (1.1).

From the second law of thermodynamics we have the relation

$$Ids = dU - \rho^{-1}(\sigma_{ij} dD^0_{ij} + \nu_{ij} dR^0_{ij}) \tag{1.9}$$

where σ_{ij} are the components of the static stress tensor and ν_{ij} are the components of the tensor thermodynamic force related to the microstructure of the medium.

We emphasize that the differential equality (1.9) relates functions defined for equilibrium states of the medium which are infinitesimally close to one another. Therefore, only the work of elastic forces related to the translational and rotational degrees of freedom is considered on the right-hand side of this relation for the viscoelastic medium with the microstructure under consideration. Equation (1.9) can be given a form which takes into account the dissipative effects explicitly if the expression for dU from (1.6) is substituted.

By (1.9) we have

$$\sigma_{ij} = \rho\left(\frac{\partial U}{\partial D_{ij}^0}\right)_s, \quad \nu_{ij} = \rho\left(\frac{\partial U}{\partial R_{ij}^0}\right)_s \tag{1.10}$$

For a purely elastic medium when there is no heat conduction, Eq. (1.9) leads to zero dissipation by virtue of (1.6) and (1.10). For a purely viscous incompressible medium the quantities in (1.10) vanish identically and the whole work of the internal forces is subject to dissipation.

As regards the components p_{ij}, π_{ij}, q_i, M_i, in the spatially-local theory with heredity their values at the point x_{i0} at time t_0 are determined by the previous history of the particle, i.e. they depend on the following functions of t_1

$$T(t_1, x_1),\ D(t_1, t_0, x_{i0}),\ G(t_1, t_0, x_{i0})$$

($t_1 \leqslant t_0$, (1.1) holds) and on the derivatives of these functions with respect to x_{i0}.

From (1.4)–(1.6), (1.9) and (1.10) we compute the rate of change of entropy

$$\rho T\frac{ds}{dt} = (p_{ij} - \sigma_{ij})\upsilon_{i,j} + \pi_{ij}\Omega_{i,j} - M_i\Omega_i - \nu_{ij}\frac{d}{dt}R_{ij}^0 + \varepsilon_{ijk}p_{jk}\Omega_i + \varepsilon - q_{i,i} \tag{1.11}$$

The viscous stress tensor with components

$$\tau_{ij} = p_{ij} - \sigma_{ij}$$

consists of the symmetric part $\tau_{ij} = \tau_{(ij)}$ and the antisymmetric part $\tau_{ij}^a = \tau_{[ij]}$. Entropy production can now be computed from (1.7) and (1.11)

$$\sigma = T^{-1}\left(\tau_{ij}^s\upsilon_{(i,j)} + \tau_{ij}^a(\upsilon_{[i,j]} + \varepsilon_{ijk}\Omega_k) + \pi_{ij}\Omega_{i,j} - M_i\Omega_i - \nu_{ij}\frac{d}{dt}R_{ij}^0\right) + q_i(T^{-1})_{,i} \tag{1.12}$$

We shall seek expressions for the dissipative terms within the framework of linear non-equilibrium thermodynamics. Thus, we adopt the following form of (1.1)

$$X_i(t_1, t_0, x_{j0}) = x_{i0} + u_i(t_0, x_{j0}) - u_i(t_1, x_{j0})$$

where the displacement vector $u_i = u_i(t, x_j)$ is small. For the rotation matrix we use the functional form

$$G_{ij}(t_1, t_0, x_{k0}) = \delta_{ij} + \varepsilon_{ijk}(\varphi_k(t_0, x_{j0}) - \varphi_k(t_1, x_{j0}))$$

where the rotation vector $\varphi_i = \varphi_i(t, x_j)$ is small. Then

$$\upsilon_i(t, x_j) = \frac{\partial}{\partial t}u_i(t, x_j),\quad \Omega_i(t, x_j) = \frac{\partial}{\partial t}\varphi_i(t, x_j) \tag{1.13}$$

The relative rotation matrix has the form

$$R_{ij}(t_1, t_0, x_{k0}) = \delta_{ij} + \varepsilon_{ijk}(\varphi_k^0(t_0, x_{j0}) - \varphi_k^0(t_1, x_{j0}))$$

$$\varphi_i^0 = \varphi_i + \frac{1}{2}\varepsilon_{ijk}u_{j,k}$$

where $\varphi_i^0 = \varphi_i^0(t, x_j)$ is the relative rotation vector.

Let us introduce some new symbols

$$e = v_{i,i},\quad e_{ij} = v_{(i,j)} - \frac{1}{3}\delta_{ij}e$$
$$\tau = \tau^s_{ii},\quad s_{ij} = \tau^s_{ij} - \frac{1}{3}\delta_{ij}\tau$$
$$\psi = \Omega_{i,i},\quad \psi_{ij} = \Omega_{(i,j)} - \frac{1}{3}\delta_{ij}\psi$$
$$\gamma = \pi_{ii},\quad \gamma_{ij} = \pi_{(ij)} - \frac{1}{3}\delta_{ij}\gamma$$
$$\nu_i = \varepsilon_{ijk}\nu_{jk},\quad \mu_i = -M_i,\quad \xi_i = \varepsilon_{ijk}\tau^a_{jk} - \nu_i$$
$$\Omega^0_i = \frac{\partial}{\partial t}\varphi^0_i,\quad \varkappa_i = -T^{-1}T_{,i}$$

Using this notation and (1.13), we transform (1.12) into

$$\sigma = T^{-1}(\tau e + s_{ij}e_{ij} + \gamma\psi + \gamma_{ij}\psi_{ij} + \mu_i\Omega_i + \xi_i\Omega^0_i + q_i\varkappa_i) \tag{1.14}$$

We assume that the quantities τ, s_{ij}, γ, γ_{ij}, μ_i, ξ_i, q_i are linear functionals of the values of the variables e, e_{ij}, ψ, ψ_{ij}, Ω^0_i, Ω_j, $\varkappa_i$ at a particle of the medium. By virtue of translational invariance these functionals must have the form of convolutions with respect to time with some relaxation kernels. For an isotropic medium, by local $SO(3)$ invariance we obtain

$$\begin{gathered} \tau = K^1_{11} * e + K^1_{12} * \psi,\quad \gamma = K^1_{21} * e + K^1_{22} * \psi \\ s_{ij} = K^2_{11} * e_{ij} + K^2_{12} * \psi_{ij},\quad \gamma_{ij} = K^2_{21} * e_{ij} + K^2_{22} * \psi_{ij} \\ \xi_i = K^3_{11} * \Omega^0_i + K^3_{12} * \Omega_i,\quad \mu_i = K^3_{21} * \Omega^0_i + K^3_{22} * \Omega_i \\ q_i = K^3 * \varkappa_i \end{gathered} \tag{1.15}$$

The relaxation kernels $K^\alpha_{AB} = K^\alpha_{AB}(t)$ vanish when $t < 0$ (causality). Moreover, they must also satisfy a number of conditions which follow from the general laws of physics. There is a standard method of obtaining such conditions [5]. If we use it, we obtain: *an analogue of Onsager's relation* $K^\alpha_{AB}(t) = K^\alpha_{BA}(t)$ (*as a consequence of reversibility at the micro level*).

The dissipation condition (*which follows from (1.7), (1.14) and (1.15)*): if $L^\alpha_{AB}(\omega)$ are the Fourier transforms of $K^\alpha_{AB}(t)$, then Re $L^\alpha_{AB}(\omega)$ is a positive definite matrix for $\alpha = 1, 2, 3$ and any real ω.

The construction of a model of a viscoelastic medium with a microstructure is thus completed, since a complete system of equations and defining relations has been formulated and the general laws of mechanics and thermodynamics are satisfied.

Further details of the model can be given if expressions for the internal energy and the relaxation kernels are specified, which must be done based on some additional information about the nature of the medium under consideration. The possibility of deriving results from general postulates has been exhausted.

We note that general defining relations for models of media with a microstructure without a memory were studied by Misicu [6].

2. We shall study the propagation of small perturbations in a homogeneous medium. To do this we need to specify an expression for the internal energy U and linearize Eqs (1.4)–(1.6). The first expression in (1.4) is not essential by virtue of (1.8). We put $\vartheta = T - T_0$.

We observe that $D^0_{ij} = \delta_{ij} - \varepsilon_{ij}$, $\varepsilon_{ij} = u_{(i,j)}$.

For U we take an expression that is quadratic in the components of the strain tensor and the angles of rotation

$$\begin{gathered} U(T, \varepsilon_{ij}, \varphi^0_i) = U_0(T) + \frac{1}{2}\rho_0^{-1}(\lambda_1(\varepsilon_{ii})^2 + \lambda_2\varepsilon_{ij}\varepsilon_{ij} + \zeta\varphi^0_i\varphi^0_i); \\ C_V = \frac{d}{dT}U_0(T_0);\quad \lambda_1, \lambda_2, \zeta > 0 \end{gathered} \tag{2.1}$$

For an arbitrary function $f = f(t, x_i)$ we shall denote by f_F the Fourier transform

$$f_F(\omega, k_i) = \int \exp(-i\omega t - ik_i x_i) f(t, x_i) dt\, dx_j$$

We linearize the system of equations (1.4)–(1.6) using (1.10), (1.15) and (2.1), and we apply a Fourier transformation. Without loss of generality we can adopt the expression $k_i = k\delta_{i1}$ for the wave vector. We obtain a homogeneous system of seven linear algebraic equations with seven unknowns u_{iF}, φ_{iF}, ϑ_F

$$\begin{aligned}
&\rho_0 i\omega v_{iF} - ik_j p_{ijF} = A_{ij}^1 u_{jF} + B_{ij}^1 \varphi_{jF} = 0\\
&J i\omega \Omega_{iF} + \varepsilon_{ijl} p_{jlF} - ik_j \pi_{ijF} - M_{iF} = A_{ij}^2 u_{jF} + B_{ij}^2 \varphi_{jF} + C_i^2 \vartheta_F = 0\\
&\rho_0 C_V i\omega \vartheta_F + ik_i q_{iF} = B_i^3 \varphi_{iF} + C^3 \vartheta_F = 0
\end{aligned} \tag{2.2}$$

Here we use the following brief notation for the coefficients

$$\begin{aligned}
&A_{ij}^1 = A^{11}\delta_{ij} + A^{12} k_i k_j\\
&B_{ij}^1 = B^{11}\delta_{ij} + B^{12} k_i k_j + B^{13}\varepsilon_{kij} ik_k\\
&A_{ij}^2 = A^{21}\delta_{ij} + A^{22} k_i k_j + A^{23}\varepsilon_{kij} ik_k\\
&B_{ij}^2 = B^{21}\delta_{ij} + B^{22} k_i k_j, \quad C_i^2 = ik_i C^2, \quad B_i^3 = ik_i B^3\\
&A^{11} = -\rho_0\omega^2 + \frac{1}{2}\left(\lambda_2 + \frac{1}{2}\zeta\right)k^2 + \frac{1}{2}\left(L_{11}^2 + \frac{1}{2}L_{11}^3\right) i\omega k^2\\
&A^{12} = \lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{4}\zeta + \left(\frac{1}{6}L_{11}^2 + L_{11}^1 + \frac{1}{4}L_{11}^3\right) i\omega\\
&B^{11} = L_{12}^2 i\omega k^2, \quad B^{12} = \left(\frac{1}{6}L_{12}^2 + L_{12}^1\right) i\omega k^2\\
&B^{13} = \frac{1}{2}((L_{11}^3 + L_{12}^3) i\omega + \zeta), \quad A^{21} = \frac{1}{2}L_{21}^2 i\omega k^2\\
&A^{22} = \left(\frac{1}{6}L_{21}^2 + \frac{1}{3}L_{21}^1\right) i\omega, \quad A^{23} = \frac{1}{2}L_{11}^3 i\omega + \frac{1}{2}\zeta\\
&B^{21} = -J\omega^2 + \zeta + (L_{11}^3 + L_{12}^3 + L_{21}^3 + L_{22}^3) i\omega + \frac{1}{2}L_{22}^2 i\omega k^2\\
&B^{22} = \left(\frac{1}{6}L_{22}^2 + \frac{1}{3}L_{22}^1\right) i\omega k^2, \quad C^2 = 0\\
&B^3 = 0, \quad C^3 = \rho_0 C_V i\omega + T_0^{-1} L_{33}^3 k^2
\end{aligned}$$

By direct computation we obtain an expression for the determinant of (2.2)

$$\begin{aligned}
&\Delta = P_1 P_2\\
&P_1 = B^{21}C^3(A^{11} + k^2 A^{12}) + (A^{11} + k^2 A^{12})(B^{22}C^3 + C^2 B^3)k^2 -\\
&-(A^{21} + k^2 A^{22})(B^{11} + k^2 B^{12})C^3\\
&P_2 = (A^{11}B^{21} - A^{21}B^{11} - k^2 A^{23}B^{13})^2 - (A^{23}B^{11} + A^{21}B^{13})^2 k^2
\end{aligned} \tag{2.3}$$

The dispersion relation $P_1 = 0$ describes the dynamics of longitudinal translational and rotational oscillations as well as heat-conduction effects. The dispersion relation $P_2 = 0$ describes the dynamics of the related transverse and rotational modes. To verify these assertions it is convenient to compute P_1, P_2 in the dissipation-free approximation on the basis of (2.3)

$$P_1 = Q_1 Q_2 Q_3, \quad P_2 = Q_4^2$$
$$Q_1 = \rho_0 C_V i\omega, \quad Q_2 = -\omega^2 J + \zeta$$
$$Q_3 = \rho_0(-\omega^2 + V_1^2 k^2), \quad V_1 = \rho_0^{-1/2}(\lambda_1 + \lambda_2)^{1/2}$$
$$Q_4 = \left(-\rho_0\omega^2 + \frac{1}{2}\left(\lambda_2 + \frac{1}{2}\zeta\right)k^2\right)(-J\omega^2 + \zeta) - \frac{1}{4}\zeta^2 k^2$$

The dispersion relations $Q_A = 0$ ($A = 1, 2, 3$) describe, respectively, the absence of heat transfer, rotational oscillations of a non-wave nature with frequency $\omega_0 = (\zeta/J)^{1/2}$, and longitudinal elastic waves with velocity V_1.

The dispersion relation

$$Q_4 = 0 \tag{2.4}$$

describes the related transverse translation-rotation waves. At small wave numbers Eq. (2.4) has four solutions for frequencies with the following asymptotic forms: $\omega = \pm\omega_0 + o(1)$ (optical modes), and $\omega = \pm V_2 k + o(k)$, $V_2 = (2\rho_0)^{-1/2}\lambda_2^{1/2}$ (acoustic modes).

3. We shall now consider the functional form of the relaxation kernels appearing in the relations for Cosserats' viscoelastic medium.

Let $K = K(t)$ be a typical kernel and let $L = L(\omega)$ be its Fourier transform. The function $L = L(\omega)$ is the boundary value of a function holomorphic in the complex half-plane Im $\omega \leqslant 0$ (the Paley–Wiener theorem [7]). When continued from the lower to the upper half-plane the function $L = L(\omega)$ will develop singularities. They correspond to internal relaxation processes taking place in the medium under consideration. A detailed description of these processes is only possible at a more fundamental level of study as compared with that of the model of a continuous medium. In the phenomenological approach used in the present paper the internal relaxation processes are characterized by the contributions introduced by $L = L(\omega)$.

In a large class of cases we have a family of exponentially decaying processes (without oscillations) with some spectrum of internal relaxation times. This corresponds to the functional form of the kernel

$$K(t) = \int_0^{+\infty} \frac{A(\chi)}{\chi} \exp\left(-\frac{t}{\chi}\right) d\chi \tag{3.1}$$

The kernel (3.1) has the following Fourier transform

$$L(\omega) = \int_0^{+\infty} \frac{A(\chi)}{1 + i\chi\omega} d\chi \tag{3.2}$$

If we assume that the inequality $A(\chi) \geqslant 0$ for the weight function is satisfied, then the following dissipation condition will be satisfied automatically for $L(\omega)$ on the real axis (see the end of Section 1): Re $L(\omega) \geqslant 0$.

There are widely used rheological models in which the weight function $A(\chi)$ is a finite sum of δ-shaped distributions, which corresponds to a finite set of internal relaxation processes. In this case the rheological relationships can be written in the functional form [8, 9]

$$H_1\left(\frac{d}{dt}\right) f = H_2\left(\frac{d}{dt}\right) z$$

where $z = z(t)$ is a certain degree of freedom of the material, $f = f(t)$ is an external force, and H_1, H_2 are polynomials.

It is of interest to find the form of the weight function $A(\chi)$ for real media. To do this one can use experiments concerning the viscoelastic behaviour of materials subject to a constant force. If the behaviour of the degree of freedom z under the action of an instantaneously applied force f_0 is considered, then for a linear viscoelastic material

$$z(t) = f_0(r_0 + r(t)), \qquad r(0) = 0 \tag{3.3}$$

where r_0 is a constant characterizing the elastic properties of the material and $r = r(t)$ is a monotone increasing function. The rheological relation

$$f_0\vartheta(t) = \lambda z + K * \dot{z} \tag{3.4}$$

holds, where λ = the corresponding modulus of elasticity.

By eliminating $z(t)$ from (3.3) and (3.4), we can express $K(t)$ in terms of $r(t)$. Indeed, we denote the Fourier transform of $r(t)$ by $R(\omega)$. Then from (3.3) and (3.4) we obtain the relation

$$1 = (\lambda + i\omega L) \ \ (r_0 + i\omega R) \tag{3.5}$$

It is obvious that $\lambda r_0 = 1$. Next, from (3.5) we obtain

$$L = -\lambda R(r_0 + i\omega R)^{-1} \tag{3.6}$$

Considering the behaviour of (3.2) near the imaginary axis and using the Sokhotskii–Plemelj formula, we derive the expression

$$L(iy + \varepsilon) - L(iy - \varepsilon) = -2\pi i y^{-1} A(y^{-1}), \ \ y > 0 \tag{3.7}$$

for the jump. Applying this formula to the right-hand side of (3.6), we can determine the weight function $A(\chi)$ and then, in principle, the kernel $K(t)$.

Geological-geodesic experimental data [10] are known to be well described by the following functions

$$r(t) = a(t/\chi_0)^\alpha, \ \ 0 < \alpha < 1; \ \ r(t) = a\ln(1 + t/\chi_0) \tag{3.8}$$

We shall study two cases in succession. We determine the Fourier transform of the first function in (3.8) using formula No. 3.381.4 in [11]

$$R(\omega) = R_0(i\omega)^{-(1+\alpha)}, \quad R_0 = a\chi_0^{-\alpha}\Gamma(1+\alpha)$$

and assuming that the cut passes along the positive imaginary half-axis. Now, using (3.6) and (3.7), we find the weight function

$$A(\chi) = (2\pi)^{-1} R_0 \sin(\alpha\pi)\chi^\alpha((r_0\cos(\alpha\pi) + R_0\chi^\alpha)^2 + (r_0\sin(\alpha\pi))^2)^{-1}$$

Thus, the internal relaxation processes are continuously distributed. The contribution of the relaxation processes with $\chi \to 0$, $\chi \to +\infty$ tends to zero as χ^α and $\chi^{-\alpha}$, respectively.

We now find the second function in (3.8). Applying formula No. 4.331.2 in [11], we obtain

$$R(\omega) = -a\exp(-i\omega\chi_0)(i\omega)^{-1}\operatorname{Ei}(-i\omega\chi_0)$$

Here $\operatorname{Ei}(z)$ is the integral exponential function. Using (3.6) and (3.7) again, we find

$$A(\chi) = a\exp(-\chi_0/\chi)(a^2\pi^2 + (r_0\exp(-\chi_0/\chi) - a\operatorname{Re}\operatorname{Ei}(\chi_0/\chi))^2)^{-1}$$

It is clear that a continuous spectrum of internal relaxation processes is also realized in the model corresponding to the second function in (3.8), the processes with long and short times being suppressed.

The above examples show that in the description of real media it is preferable to use relaxation kernels having a continuous spectrum of internal relaxation times.

This research was supported by the International Science Foundation (NP1000 and NP1300).

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Translated by T.J.Z.